MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

EARTH COVERAGE ("Footprint") OF A SATELLITE-BORNE ANTENNA

S. L. ZOLNAY

Group 63

TECHNICAL NOTE 1971-7

5 FEBRUARY 1971

LEXINGTON MASSACHUSETTS

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the Department of the Air Force under Contract F19628-70-C-0230.

ABSTRACT

In this note an exact solution is given to the problem of the intersection of an arbitrarily oriented cone and a sphere for the application of finding the Earth coverage--footprint--of a narrow beam satellite borne antenna. Two cases are considered: when the pointing angles from the satellite are known, and when the coordinates of the point on the Earth toward which the boresight is aimed are known. The situation of partial coverage is also treated in detail. Examples are included to illustrate the method.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col., USAF
Chief, Lincoln Laboratory Project Office

List of Symbols

α = vertex angle of cone

$\underline{R}$ = range vector from satellite to Earth

p = pitch angle

ρ = roll angle

$\underline{i}$, $\underline{j}$, $\underline{k}$, $\underline{i}'$, $\underline{I}$, etc. = unit vectors in respective coordinate systems

δ = instantaneous declination of satellite

$\underline{r}$ = radius vector of Earth

C = curve of intersection between cone and sphere

P = piercing point of boresight axis

P_n = piercing point of M_n vector

m = a numeric

ϕ' = longitude separation of a point from subsatellite longitude

$\underline{u}$ = an auxiliary vector used in defining surface of cone and vectors M_n

$\underline{M}_n$ = a vector on surface of cone

β = an auxiliary angular parameter used in locating $\underline{M}_n$

b, c, d = angles of triangle formed by $\underline{i}$, $\underline{M}_n$, and $m\underline{r}$

M_n = length of $\underline{M}_n$ ($M_n \neq |\underline{M}_n|$)

τ = tilt angle of plane of triangle relative to yz -plane

EARTH COVERAGE ("FOOTPRINT")
OF A SATELLITE-BORNE ANTENNA

I. INTRODUCTION

The purpose of this Technical Note is to present an exact solution for the problem of the intersection of an arbitrarily oriented cone and sphere. The motivation for considering this problem is to provide an aid for (a) evaluating the coverage--footprint--on the surface of the Earth of a satellite-borne antenna, and (b) assessing the threats of jamming and of eavesdropping. Vectorial and trigonometric methods are used to obtain the solution. First, an expression is derived for computing the latitude and longitude coordinates of the point on the Earth where the boresight axis of the antenna pierces the surface of the sphere. Next, a chosen number of vectors are generated around this axis such that all lie on the surface of the cone. Another expression is derived for computing the coordinates of those points where each of these vectors pierces the surface of the sphere. A computer program was written for solving these expressions. The outputs of the program are a listing of the coordinates of the points of intersection and a graphical plot of these points. Two examples are included to illustrate the method.

II. THE INTERSECTION OF AN ARBITRARILY ORIENTED CONE AND A SPHERE

Consider two coordinate systems: OXYZ and Oxyz. A sphere of radius r has its center at the origin of the first one, a cone of vertex angle α has its vertex at the origin of the second one. The orientation of one coordinate system relative to the other may be arbitrary; however, their instantaneous relationship is known. The origins are separated by a distance $(m)(r)$ measured along Oy, where m is a numeric and it is greater than unity for orbiting satellites. (The simple problem of intersection of a cone and a sphere when the vertex of the cone is within the sphere is not treated here.) The range vector $\underline{R}$ --which is also the axis of the right circular cone--from the origin of Oxyz to the point on the Earth is known. Given that a cone of vertex angle α is described around axis $\underline{R}$, one would like to

know the curve C of the first intersection of this cone with the sphere. (The intersection of the cone emerging from the sphere is of no interest.) The problem is illustrated in Fig. 1.

The range vector, $\underline{R}$, is given in the Oxyz coordinate system:

$$\underline{R} = \underline{i}\, R_i + \underline{j}\, R_j + \underline{k}\, R_k \tag{1}$$

where $\underline{i}$, $\underline{j}$, $\underline{k}$ are unit vectors. Define pitch and roll angles, p and ρ, as shown in Fig. 1:

$$\sin p = \frac{R_i}{\sqrt{R_i^2 + R_j^2}} \quad , \quad \text{and} \tag{2}$$

$$\tan \rho = \frac{R_K}{\sqrt{R_i^2 + R_j^2}} \quad . \tag{3}$$

The $\underline{R}$ vector can be written in terms of these angles:

$$\underline{R} = \underline{i}\,(|\underline{R}| \cos \rho \sin p) + \underline{j}\,(|\underline{R}| \cos \rho \cos p) + \underline{k}\,(|\underline{R}| \sin \rho) \tag{4}$$

where $|\underline{R}|$ is the magnitude of $\underline{R}$. Let Oxyz be rotated around Ox, in the direction indicated by the arrow in Fig. 1, by the angle δ, the declination angle of the satellite, resulting in the primed coordinate system. The range vector is

$$\begin{aligned} \underline{R}' = {} & \underline{i}'(|\underline{R}| \cos \rho \sin p) \\ & + \underline{j}'[(|\underline{R}| \cos \rho \cos p)\cos \delta - (|\underline{R}| \sin \rho \sin \delta] \\ & + \underline{k}'[(|\underline{R}| \cos \rho \cos p)\sin \delta + (|\underline{R}| \sin \rho)\cos \delta] \end{aligned} \tag{5}$$

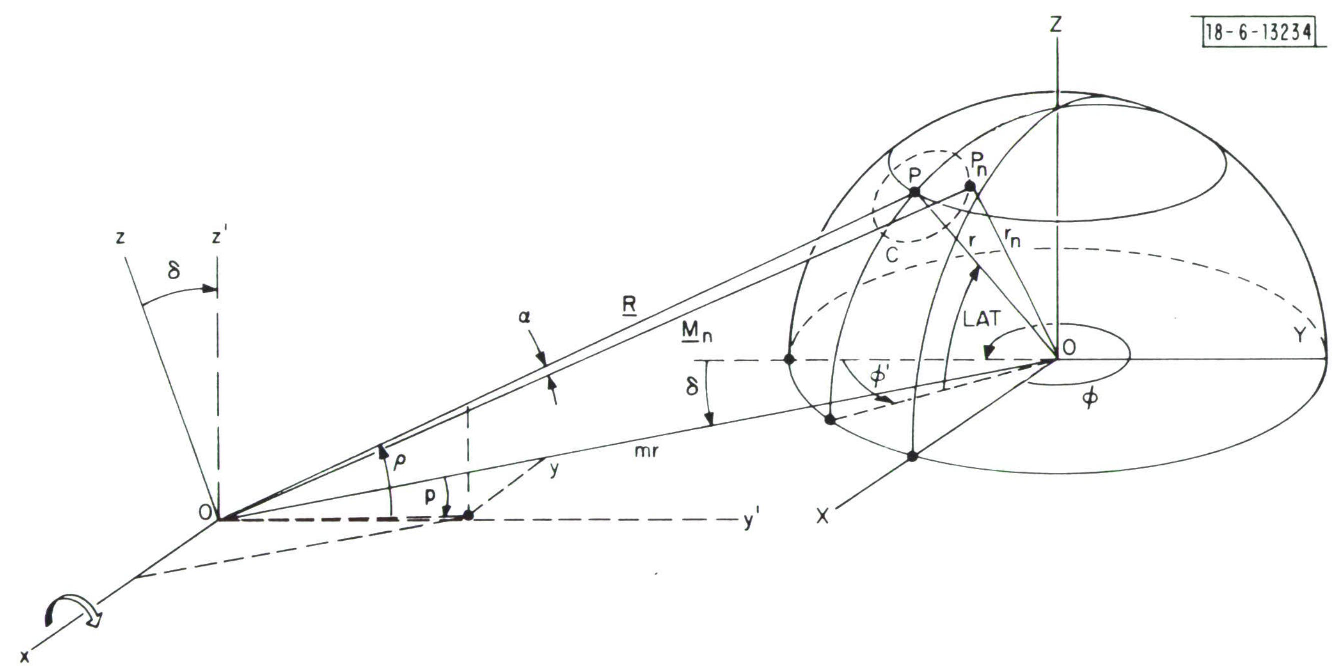

Fig. 1. Intersection of a cone and a sphere.

which expression reduces to Eq. 4 for $\delta = 0$. The new expressions for the pitch and roll angles are

$$\sin p' = \frac{R_{i'}}{\sqrt{R_{i'}^2 + R_{j'}^2}} \quad , \quad \text{and} \tag{6}$$

$$\tan \rho' = \frac{R_{k'}}{\sqrt{R_{i'}^2 + R_{j'}^2}} \quad . \tag{7}$$

The radius vector $\underline{r}$ to the piercing point, P is

$$\begin{aligned} \underline{r} = \ & \underline{I}(|\underline{R}| \cos \rho' \sin p') \\ & - \underline{J}(m|\underline{r}| \cos \delta - |\underline{R}| \cos \rho \ \cos p') \\ & + \underline{K}(|\underline{R}| \sin \rho' - m|\underline{r}| \sin \delta) \end{aligned} \tag{8}$$

where $|\underline{r}| = 6378.16$ kilometers. The coordinates of P are

$$\tan \text{latitude} = \frac{r_K}{\sqrt{r_I^2 + r_J^2}} \quad , \quad \text{and} \tag{9}$$

$$\sin \phi' = \frac{r_I}{\sqrt{r_I^2 + r_J^2}} \tag{10}$$

The longitude (East) of P follows from Eq. 10 and from the longitude (East), ϕ, of the sub-satellite point:

$$\text{Longitude (East)} = \phi + \phi' \tag{11}$$

In Fig. 1 the coordinates have been arbitrarily aligned such that the sub-satellite longitude is 270°E .

Having computed the latitude and longitude of the piercing point of $\underline{R}$, one would like to form an arbitrary number of vectors all lying on the surface of the cone and similarly compute their piercing points. Let the coordinate system $Ox'y'z'$ be rotated by the angle p' around the axis OZ', then the resultant system be further rotated by ρ' around the resultant x axis. The transformation is:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos p' & \sin p' & 0 \\ -\sin p' & \cos p' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \rho' & -\sin \rho' \\ 0 & \sin \rho' & \cos \rho' \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \tag{12}$$

In the doubly-primed system let there be a vector $\underline{u}$ as illustrated by Fig. 2. The magnitude of $\underline{u}$ is

$$|\underline{u}| = |\underline{R}| \tan \alpha \quad , \tag{13}$$

and the instantaneous value of $\underline{u}$ is

$$\underline{u} = i'' \; |\underline{u}| \cos \beta + k'' \; |\underline{u}| \sin \beta \tag{14}$$

where β is a convenient angular parameter, measured clockwise from the $x''y''$-plane. The $n^{\underline{th}}$ $\underline{M}$ vector, $\underline{M}_n$, lying on the surface of the cone is defined as

$$\underline{M}_n = \underline{R} + \underline{u}_n \tag{15}$$

which can be written in the single-primed coordinate system as

$$\begin{aligned} \underline{M}'_n = \sqrt{R_{i'}^2 + R_{j'}^2 + R_{k'}^2} \Big[& \underline{i}'(\cos \rho' \sin p' + \tan \alpha \cos \beta_n \cos p' - \tan \alpha \sin \beta_n \sin \rho' \sin p') \\ & + \underline{j}'(\cos \rho' \cos p' - \tan \alpha \cos \beta_n \sin p' - \tan \alpha \sin \beta_n \sin \rho' \cos p') \\ & + \underline{k}'(\sin \rho' + \tan \alpha \sin \beta_n \cos \rho') \Big] \quad ; \end{aligned} \tag{16}$$

and in the Oxyz coordinate system as

$$\underline{M}_n = \sqrt{R_{i'}^2 + R_{j'}^2 + R_{k'}^2} \Big\{ \underline{i}(\cos \rho' \sin p' + \tan \alpha \cos \beta_n \cos p' - \tan \alpha \sin \beta_n \sin \rho' \sin p')$$

$$+ \underline{j} \Big[(\cos \rho' \cos p' - \tan \alpha \cos \beta_n \sin p' - \tan \alpha \sin \beta_n \sin \rho' \cos p') \cos \delta$$

$$+ (\sin \rho' + \tan \alpha \sin \beta_n \cos \rho') \sin \delta \Big]$$

$$+ \underline{k} \Big[(\cos \rho' \cos p' - \tan \alpha \cos \beta_n \sin p' - \tan \alpha \sin \beta_n \sin \rho' \cos p')(-\sin \delta)$$

$$+ (\sin \rho' + \tan \alpha \sin \beta_n \cos \rho') \cos \delta \Big] \Big\} \quad . \tag{17}$$

From Fig. 2 the following trigonometric relations can be written:

$$\cos b_n = \frac{M_{nj}}{\sqrt{M_{ni}^2 + M_{nj}^2 + M_{nk}^2}} \tag{18}$$

$$\sin c_n = m \sin b_n \tag{19}$$

$$d_n = 180^\circ - c_n - b_n \tag{20}$$

$$M_n^2 = |\underline{r}|^2 + m^2 |\underline{r}|^2 - 2\, m |\underline{r}|^2 \cos d_n \tag{21}$$

In Eq. 19 c_n can have two values: $c_{n1} < 90^\circ$ and $c_{n2} = 180^\circ - c_{n1} > 90^\circ$; normally, c_{n2} should be used only (unless one is interested in the intersection of the sphere and the cone emerging from the sphere). The expression for pitch and roll angles associated with $\underline{M}_n'$ are

$$\sin p_n = \frac{M_{ni'}}{\sqrt{M_{ni'}^2 + M_{nj}^2}} \quad , \quad \text{and} \tag{22}$$

$$\tan \rho_n = \frac{M_{nk'}}{\sqrt{M_{ni'}^2 + M_{nj}^2}} \tag{23}$$

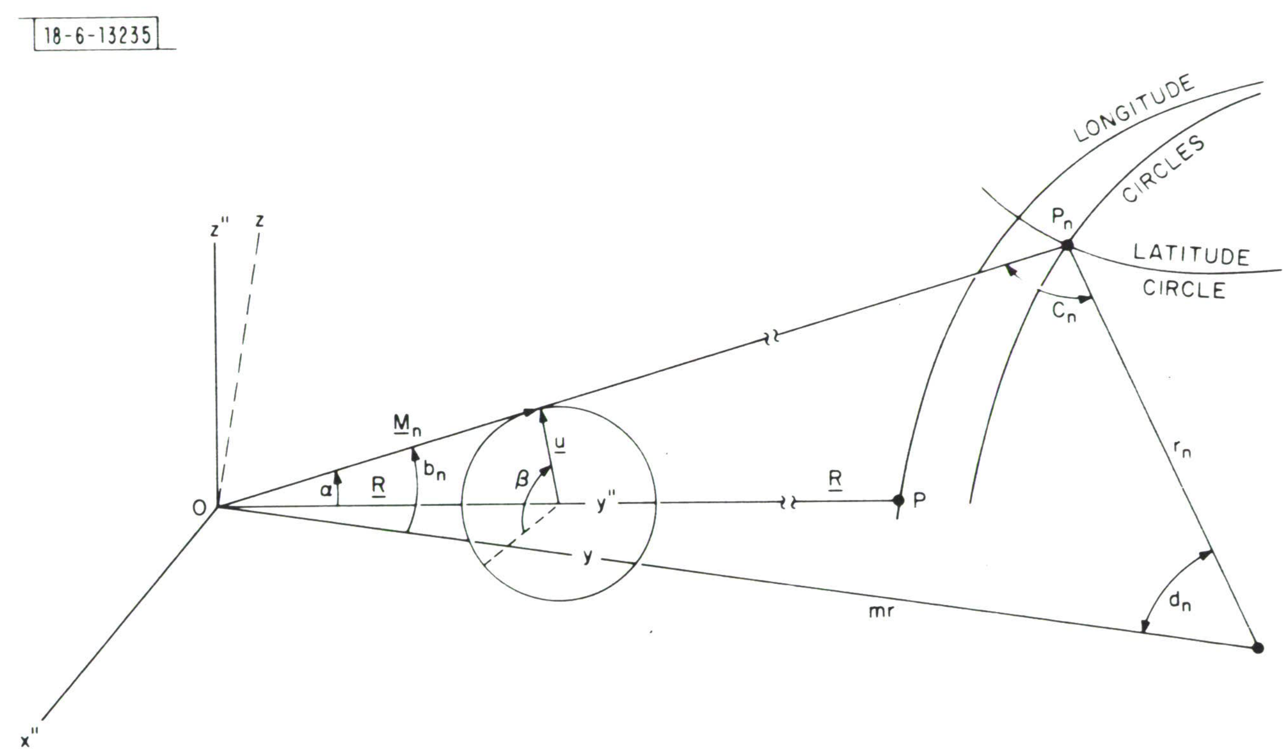

Fig. 2. Illustration of the geometry of the problem.

The radius vector $\underline{r}_n$ to the piercing point of $\underline{M}_n$ is

$$\underline{r}_n = \underline{I}(M_n \cos \rho_n \sin p_n) - \underline{J}(m|\underline{r}| \cos \delta - M_n \cos \rho_n \cos p_n) + \underline{K}(M_n \sin \rho_n - m|\underline{r}| \sin \delta) \tag{24}$$

where M_n is obtained from Eq. 21, $M_n \neq |\underline{M}_n|$ of Eq. 17. The coordinates of P_n are

$$\tan \mathrm{Lat}_n = \frac{r_{nK}}{\sqrt{r_{nI}^2 + r_{nJ}^2}} \tag{25}$$

$$\sin \phi_n' = \frac{r_{nI}}{\sqrt{r_{nI}^2 + r_{nJ}^2}} \tag{26}$$

$$\mathrm{Longitude}_n \ (\mathrm{East}) = \phi + \phi' \tag{27}$$

Equations (25) and (26) yield the latitude and longitude coordinates of the point P_n where the vector $\underline{M}_n$, which is at an angle α to the vector $\underline{R}$ and has the same origin, pierces the surface of the sphere. The $\underline{M}_n$ vectors all lie on the surface of the cone, their piercing points on the sphere are all points on the curve of first intersection. By suitably choosing values of β_n, the latitude and longitude coordinates of any point on this curve C can be readily computed; hence, C can be described with any desired degree of precision. Many antenna power patterns are circularly symmetrical, this solution to the cone-sphere intersection problem is directly applicable in these instances. The method presented here could be readily extended to other antenna pattern shapes and cross-sections such as ellipses. All that is needed is the replacement of $\underline{u}$ --which is but a parametric equation of the circle (a conic section) with β as the parameter--with a suitable parametric expression describing the shape of the cross-section of the antenna pattern.

A practical variation of the problem presented above is the situation when the range vector is unknown but the latitude and longitude of the spot P is given toward which the antenna boresight is directed. In that case Eq. 4 is replaced with

$$\begin{aligned} \underline{R} = {} & \underline{i}(|\underline{r}| \cos \text{Lat} \sin \phi') \\ & + \underline{j}\Big[(m|\underline{r}| \cos \delta - |\underline{r}| \cos \text{Lat} \cos \phi') \cos \delta \\ & \qquad + (m|\underline{r}| \sin \delta + |\underline{r}| \sin \text{Lat}) \sin \delta\Big] \\ & + \underline{k}\Big[(m|\underline{r}| \cos \delta - |\underline{r}| \cos \text{Lat} \cos \phi')(-\sin \delta) \\ & \qquad + (m|\underline{r}| \sin \delta + |\underline{r}| \sin \text{Lat}) \cos \delta\Big] , \end{aligned} \tag{28}$$

and Eq. 5 is replaced with

$$\begin{aligned} \underline{R}' = {} & \underline{i}'(|\underline{r}| \cos \text{Lat} \sin \phi') \\ & + \underline{j}'(m|\underline{r}| \cos \delta - |\underline{r}| \cos \text{Lat} \cos \phi') \\ & + \underline{k}'(m|\underline{r}| \sin \delta + |\underline{r}| \sin \text{Lat}) . \end{aligned} \tag{29}$$

The solution proceeds as above with the aid of Eqs. 6 through 27. The first results, the coordinates of the piercing point calculated with the aid of Eqs. 9 - 11 will produce the same values which were given for the coordinates of the spot P. The subsequent equations will yield coordinates of points P_n on the curve of intersection C.

In the preceeding discussion it was tacitly assumed that the first intersection was on the surface of the sphere in its entirety. Let us next consider the question of coverage. The subtended angle, b_{max}, of the Earth from the origin of Oxyz is

$$\sin(b_{max}) = \frac{1}{m} \tag{30}$$

One of the direction cosines of the range vector is

$$\cos b = \frac{R_j}{\sqrt{R_i^2 + R_j^2 + R_k^2}} \quad . \tag{31}$$

There are three possibilities: (a) Piercing point of boresight is on surface of the Earth, at least one-half of antenna beam illuminates the Earth $[b < b_{max}]$; (b) Range vector is tangent to surface of the Earth, at most one-half of antenna beam illuminates the Earth $[b = b_{max}]$; (c) Piercing point of boresight is not on surface of Earth, limited coverage is still possible $[b > b_{max}]$. These statements are valid under the assumption that the vertex angle of the cone is about the same as the angle subtended by the Earth as seen from the position of the satellite. In case of the $\underline{M}_n$ vectors on the surface of the cone, the case of interest here is when $b_n > b_{max}$ for some value of b_n. In that case, one is interested in the latitude and longitude coordinates of the points on the curve of last contact between the sphere and some inside portion of the cone. The computations proceed as follows. Replace Eq. 24 with

$$\begin{aligned} \underline{r}_n = |\underline{r}| \Big\{ & \underline{I}(\cos b_{max} \sin \tau_n) \\ & + \underline{J}(-\cos\delta \sin b_{max} - \sin\delta \cos b_{max} \cos\tau_n) \\ & + \underline{K}(-\sin\delta \sin b_{max} + \cos\delta \cos b_{max} \cos\tau_n) \Big\} \end{aligned} \tag{32}$$

and continue with Eqs. 25 through 27 to find the coordinates of the points on the curve of last contact. In Eq. 32 the tilt angle, τ, defined positive when measured counter-clockwise from yz-plane (towards yx-plane), between the yz-plane and the plane of the triangle formed by the vectors $\underline{r}_n$, $m\underline{r}$, $\underline{M}_n$ is

$$\cos\tau_n = \underline{i} \cdot \left(\frac{\underline{j} \times \underline{M}_n}{|\underline{j} \times \underline{M}_n|} \right) = \left(\frac{M_{nk}}{\sqrt{M_{ni}^2 + M_{nk}^2}} \right) \quad . \tag{33}$$

This completes considerations of the general problem of intersection of a cone and a sphere with the exception of the trivial case when $b_n > b_{max}$ for all values of b_n. In this last case there is no intersection and either the entire hemisphere is illuminated or no portion of the surface of the Earth is illuminated.

III. EXAMPLES

As an illustration of the preceeding, consider the following examples.

Subsatellite Longitude, ϕ	270°E
Cone vertex angles, α	0.6, 1.0, 1.2°
Increments of angular parameter β	10°
Coordinates of piercing point of boresight	288.733°E 42.462°N
Distance to satellite from center of Earth, m_r	6.611 Earth radii
Instantaneous declination, δ	2.0°

The results of the computation are shown graphically in Fig. 3. In the second example, shown in Fig. 4, the same parameters are used as above except $mr = 19.832$ Earth radii, and $\beta = 1^\circ$.

IV. ACKNOWLEDGMENTS

The competent assistance of Mr. F. S. Zimnoch with the computer programming is appreciated.

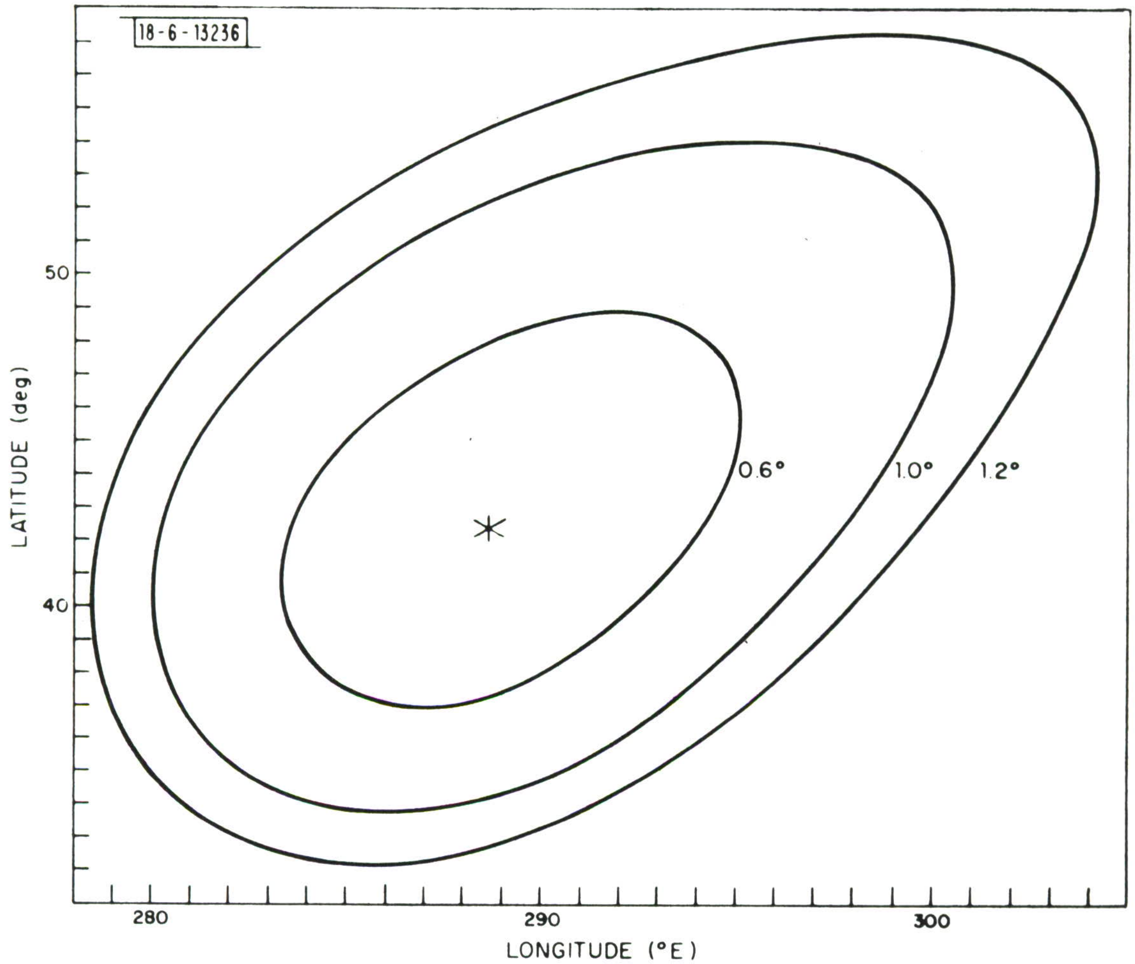

Fig. 3. Footprint of a narrow beam antenna from synchronous orbit.

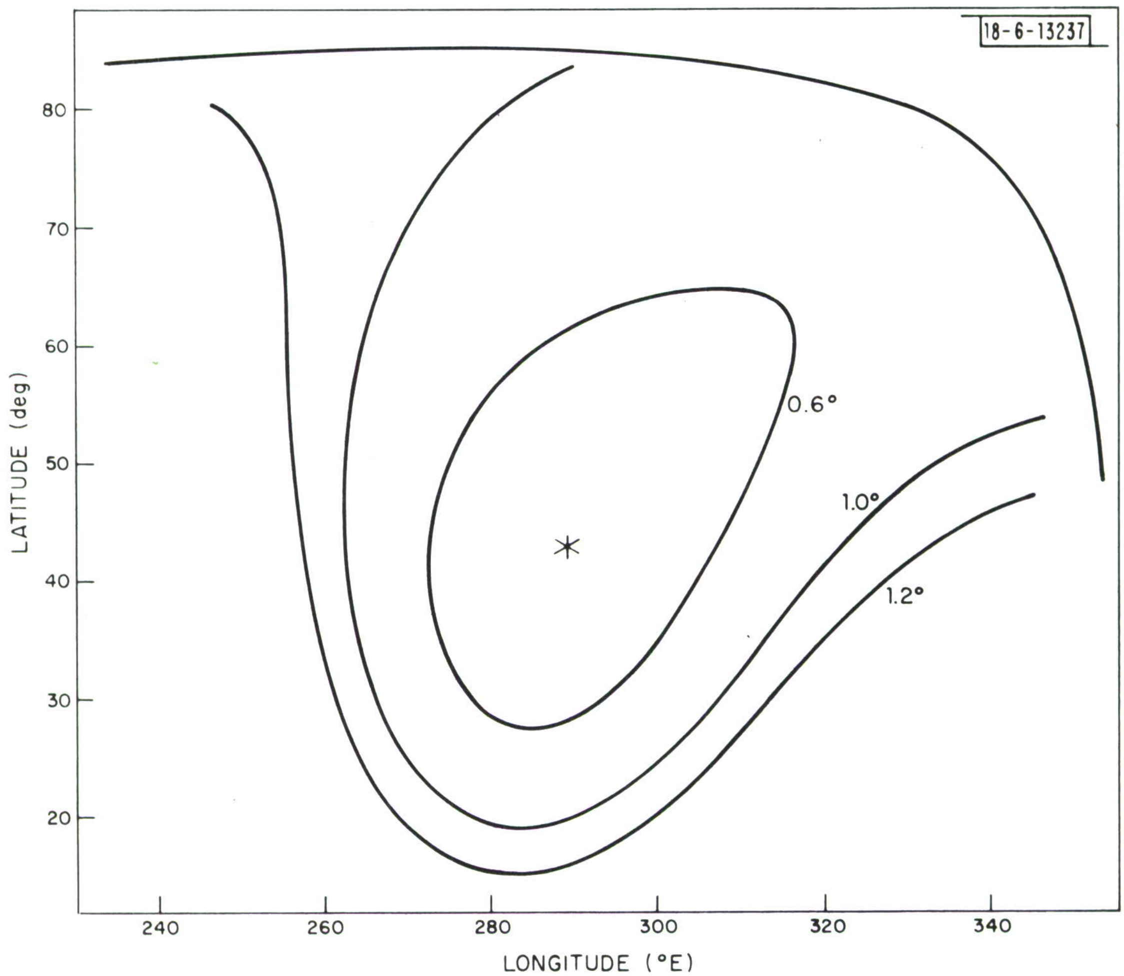

Fig. 4. Footprint of a narrow beam antenna from a distance of 19 Earth radii.

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

Earth Coverage ("Footprint") of a Satellite-Borne Antenna

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Note

5. AUTHOR(S) *(Last name, first name, initial)*

Zolnay, Stephen L.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
5 February 1971	18	0

8a. CONTRACT OR GRANT NO. F19628-70-C-0230	9a. ORIGINATOR'S REPORT NUMBER(S) TN 1971-7
b. PROJECT NO. 649L c. d.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-71-12

10. AVAILABILITY/LIMITATION NOTICES

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Air Force Systems Command, USAF

13. ABSTRACT

In this note an exact solution is given to the problem of the intersection of an arbitrarily oriented cone and a sphere for the application of finding the Earth coverage – footprint – of a narrow beam satellite borne antenna. Two cases are considered: when the pointing angles from the satellite are known, and when the coordinates of the point on the Earth toward which the boresight is aimed are known. The situation of partial coverage is also treated in detail. Examples are included to illustrate the method.

14. KEY WORDS

pointing angles	satellite
radius vector	satellite-borne antenna
range vector	